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THE CENTRAL MATHEMATICAL PROBLEM

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SUMMARY

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Symbols are introduced to distinguish various activities, items, the assumed constant flows and costs (or profits) per unit level of activity, the activity levels and the quantities of demand or availability of various items. The central problem is then stated in standard algebraic form.	
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It is shown that the problem of minimizing a linear form where the unknowns satisfy a system of equations in non-negative variables is equivalent to one where the variables satisfy a system of linear inequalities.	
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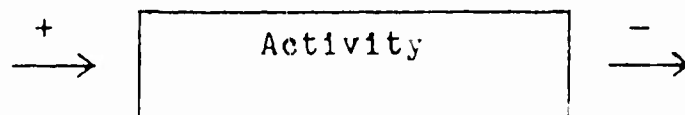
I. Algebraic Statement of the L. P. Model

The minimization of a linear form subject to linear inequality restraints has been called the central mathematical problem of linear programming. The standard form for such problems, because it arises naturally in many applications, is finding a solution of a system of linear equations in non-negative variables which minimizes a linear form. We shall see in a moment why this particular form was chosen as standard. At the same time we shall formalize in mathematical terms our remarks regarding linear programming models.

Standard Form: If the subscript $j = 1, 2, \dots, n$ denotes the j -th type of activity and x_j its quantity (or activity level), then usually $x_j \geq 0$. If, for example, x_j represents the quantity of a stockpile allocated for the j -th use, it does not, as a rule, make sense to allocate a negative quantity. In certain cases, however, one may wish to interpret a negative quantity as meaning taking stock from the j -th use. Here some care must be exercised; for example, there may be costs, such as transportation charges, which are positive regardless of the direction of flow of the stock. One must also be careful not to overdraw the stock of the using activity. For these reasons it is better in formulating models to distinguish two activities, each with a non-negative range, for their respective x_j , rather than to try incorporating them into a single range.

The interdependencies between various activities arise because all practical programming problems are circumscribed by commodity limitations of one kind or another. The limited commodity may be raw materials, manpower, facilities, or funds; these are referred to by the general term item. In chemical equilibrium problems where molecules of different types play the role of activities, the different kinds of atoms in the mixture are the items. The different types of items are denoted by a subscript i , ($i = 1, 2, \dots, m$).

In linear programming work, the quantity of an item required by an activity is usually assumed to be proportional to the quantity of activity level; or if the item is not required but produced, it is again usually assumed to be proportional to the quantity (or level) of the activity and the coefficient of proportionality is denoted by a_{ij} . The sign of a_{ij} depends on whether the item is required or produced by the activity. The sign convention used will be (+) if required and (-) if produced.



Finally, if b_i (if plus) denotes the quantity of the i^{th} item made available to the program from outside (or exogenous) sources, or (if minus) denotes the quantity required to be produced by the program, then the interdependencies between the x_j can be expressed as a set of m linear equations; the i^{th} such equation gives a complete accounting of the i^{th} item. Thus

$$(1) \quad a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

where

$$(2) \quad x_j \geq 0 \quad j = (1, 2, \dots, n).$$

Any set of values x_j satisfying (1) and (2) is called a feasible solution because the corresponding schedule is possible or feasible.

The objective of a program in practice often is the most difficult to express in mathematical terms. There are many historical reasons for this which go beyond the scope of this course. In many problems, however, the objective is simply one of carrying out the requirements (expressed by those b_i which are negative) in such a manner that total costs are minimum. Costs may be measured in dollars or in number of people involved, or the quantity of a scarce commodity used. In linear programming the total costs, denoted by z , are assumed to be a linear function of the activity levels:

$$(3) \quad c_1x_1 + c_2x_2 + \dots + c_nx_n = z .$$

The linear form z is called the objective function. In some problems the linear objective form is to be maximized rather than minimized. For example, the problem may be to produce the maximum dollar value of products under a fixed budget, fixed machine capacity, and fixed labor supply. Suppose the linear form ex-

pressing total profits to be maximized is

$$p_1x_1 + p_2x_2 + \dots + p_nx_n .$$

This is obviously mathematically equivalent to minimizing

$$-p_1x_1 - p_2x_2 - \dots - p_nx_n .$$

For these reasons the standard form of the linear programming problem is taken as the determination of a solution of a system of linear equations in nonnegative variables which minimizes a linear form.

II. Equivalent Systems

Any problem involving a system of linear inequalities can be transformed into another system in standard form by one of several devices. Steps (i) and (ii) below constitute one method, the easiest one, of accomplishing this. A second method is given by steps (i) and (ii)-alternative:

(i) Replace any linear inequality restraint such as

$$(4) \quad a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$$

by adding a slack variable $x_{n+1} \geq 0$ such that

$$(5) \quad a_1x_1 + a_2x_2 + \dots + a_nx_n + x_{n+1} = b$$

(ii) Replace any variable x_j not restricted in sign by the difference of two nonnegative variables*

*Any number can be written as the difference of two positive numbers.

$$(6) \quad x_j = x_j' - x_j'' \quad x_j' \geq 0, x_j'' \geq 0$$

$j = 1, 2, \dots, n.$

(11) Alternative — Let x_j be any variable not restricted in sign that appears in the k -th equation with a non-zero coefficient. Solve the equation for x_j and substitute its value in the remaining equations and the objective form z . Setting the k -th equation aside, the remaining modified equations constitute a reduced system of constraints. The procedure is repeated with the new linear programming problem until either:

(a) a reduced system of constraints is obtained in which all remaining variables are nonnegative, or

(b) there are no equations in the reduced system.

Once a solution to the reduced problem is obtained, a solution to the original problem is gotten by successive substitutions, in reverse order, in the eliminated equations.

Example 1: Transform the system into standard form

$$(6.1) \quad \begin{aligned} x_1 + x_2 &\geq 6 \\ x_1 + 2x_2 &= z \end{aligned}$$

Step 1: Introduce slack variable x_3

$$(6.2) \quad \begin{aligned} x_1 + x_2 - x_3 &= 6 & x_3 &\geq 0 \\ x_1 + 2x_2 &= z \end{aligned}$$

Step 2: Substitute $x_1 = x_1' - x_1''$, $x_2 = x_2' - x_2''$,

$$(6.3) \quad (x_1' - x_2'') + (x_2' - x_2'') - x_3 = 6 \quad (x_3 \geq 0; x_j' \geq 0, x_j'' \geq 0, j=1,2)$$

$$(x_1' - x_2'') + 2(x_2' - x_2'') = z.$$

Step 3: Alternative: Solve the equation $x_1 + x_2 - x_3 = 6$ for x_1 , which is unrestricted in sign,

$$(6.4) \quad x_1 = 6 - x_2 + x_3 \quad (x_3 \geq 0)$$

and substitute in the objective form z to get

$$(6.5) \quad x_2 + x_3 + 6 = z \quad (x_3 \geq 0).$$

This is case (b). A general solution to the original system (6.1) can be obtained by choosing any value for $x_3 \geq 0$, any value for x_2 ; substituting these values in (6.4) determines x_1 . Notice that no finite lower bound for z exists since x_2 may be chosen arbitrarily.

Example 2: Transform the system into standard form

$$(6.6) \quad \begin{aligned} -x_1 - x_2 &\leq -6 \\ -x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &= z \end{aligned}$$

Step 1: Introduce slack variables x_3 and x_4

$$(6.7) \quad \begin{aligned} -x_1 - x_2 + x_3 &= -6 \quad (x_3 \geq 0, x_4 \geq 0) \\ -x_1 + x_2 - x_4 &= 5 \\ x_1 + 2x_2 &= z \end{aligned}$$

Step 2: Alternative: Solve the first equation for x_1 and substitute in the second equation and the form z . Next, solve the modified second equation for x_2 and substitute in the form z . This eliminates the constraint equations and we are left with

$$(6.8) \quad z = 1/2(23 + 3x_3 + x_4) \quad (x_3 \geq 0, x_4 \geq 0),$$

and the eliminated equations

$$(6.9) \quad \begin{aligned} x_1 &= 6 - x_2 + x_3 \\ x_2 &= 1/2(11 + x_3 + x_4) \end{aligned}$$

A general solution to the original system of constraints is obtained by selecting any $x_3 \geq 0$, $x_4 \geq 0$, and determining x_2 and x_1 from (6.9). If the objective is to minimize z , then the optimum solution is found by setting $x_3 = 0$, $x_4 = 0$, obtaining $z = 23/2$, $x_2 = 11/2$, $x_1 = 1/2$.

Conversely, any problem involving equations can be replaced by an equivalent system involving only linear inequality restraints. The rule is to replace any equation

$$(7) \quad a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

by the two inequalities

$$(8) \quad \begin{aligned} a_1x_1 + a_2x_2 + \dots + a_nx_n &\geq b \\ a_1x_1 + a_2x_2 + \dots + a_nx_n &\leq b \end{aligned}$$

III. Properties of Solutions and the Simplex Method

We shall now state two important properties of solutions of a system in standard form:

Theorem 1: If a feasible solution exists for a system of m equations in nonnegative variables, then one exists in which at most m variables have positive values, the remaining variables (if any) being zero.

Theorem 2: If feasible solutions exist and possess a finite lower bound for the linear objective form z, then an optimizing solution exists in which at most m variables have positive values.

It will be convenient to use the phrase "a solution involving k variables" to mean a solution in which the remaining $n - k$ variables are zero.

Proofs of these two theorems will be given in a later lecture. Our purpose now is to illustrate these two theorems with a simple example. Consider the system

$$(8.1) \quad x_1 - 3x_2 + 4x_3 = 2 \quad (x_1 \geq 0, x_2 \geq 0, x_3 \geq 0)$$

$$(8.2) \quad x_1 + 2x_2 - x_3 = 2$$

where it is required to choose x_j such that the form z below is minimized:

$$(9) \quad x_1 + x_2 + x_3 = z .$$

We observe that if in (8.1) and (8.2) we set

$x_1 = x_2 = x_3 = 1$ we obtain a feasible solution where the cost is $z = 3$. According to the first theorem we should be able to find a feasible solution involving no more than two variables.

Moreover, a solution involving no more than two variables

exists which minimizes the value of z . This, however, is not a direct consequence of the second theorem, because the second theorem states that this is the case only if z has a finite lower bound. (This is a hedge against the possibility that values of $x_j \geq 0$ can be chosen which satisfy the constraints and make z less than any pre-assigned number, in which case, there is no minimal feasible solution.) However, for the particular example, it is clear that always

$$(10) \quad z = x_1 + x_2 + x_3 \geq 0$$

and the second theorem also applies.

Let us list the three possible solutions to (8.1) and (8.2) involving only two of the three variables. These are:

1st solution (optimal)	$x_1 = 2$	all others (in this case x_3) = 0
	$x_2 = 0$	
	$z = 2$	
2nd solution (optimal)	$x_1 = 2$	all others (in this case x_2) = 0
	$x_3 = 0$	
	$z = 2$	
3rd solution	$x_2 = 2$	all others (in this case x_1) = 0
	$x_3 = 2$	
	$z = 4$	

all three of the combinations turned out in this case to be feasible solutions. Moreover, the first and second solutions are identical. By the second theorem, the first (and second) solution must be an optimal solution to the original problem. We shall now give a direct proof that this is the case.

Testing Optimality of a Solution: Indeed, the first solution is optimal because if we multiply (8.1) by $1/5$ and (8.2) by $4/5$ and subtract from the minimizing form (9), we will eliminate the variables x_1 and x_2 and will be left with

$$(11) \quad x_3 + 2 = z,$$

and it is clear that $x_3 \geq 0$ implies $z \geq 2$, but a value of 2 can be achieved by setting $x_1 = 2$, $x_2 = 0$, $x_3 = 0$. Hence, this is a minimizing solution.

Problem: Show by eliminating the variables x_1 and x_3 from the objective form that the second solution is optimal also.

Improving a Solution: On the other hand, our third solution is not optimal, because if we multiply (8.1) by $3/5$ and (8.2) by $7/5$ and subtract from (9), we will eliminate the variables x_2 and x_3 from the minimizing form and will be left in this case with

$$(12) \quad -x_1 + 4 = z,$$

hence it pays to increase the value of x_1 , if possible. If we do so, this will affect the values of the selected variables x_2 and x_3 : they will become

$$x_2 = 2 - x_1$$

$$x_3 = 2 - x_1.$$

It follows that x_1 may be increased to 2 before x_2 or x_3 becomes negative and this will be an improved (in fact, optimal) solution. The new feasible solution may be examined for optimality in the same manner. If it were not optimal it could be used in turn to construct a third lower cost solution.

The procedure just outlined is called the simplex method and is the one in general use for solving linear programming problems.

IV: Existence of Solutions, Uniqueness

Just as in the special case of solving linear equations, it is possible that there exist no solutions to a system of linear inequalities, or there may exist many. To see this geometrically, let us take the linear programming problem in the form of a system of linear inequality restraints instead of standard form. Consider the system

$$\begin{aligned} (13) \quad & x_1 + x_2 \geq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & + x_1 - x_2 \geq 3 \end{aligned}$$

Let (x_1, x_2) represent the cartesian coordinates of a point in a plane. All points (x_1, x_2) that satisfy $x_1 + x_2 \geq 2$ lie on one side of the line $x_1 + x_2 = 2$. Which side can be determined by substituting the coordinates of some fixed point (such as

the origin) into the linear inequality. If the linear inequality is satisfied, all such points are on the same side as the fixed point; otherwise, on the opposite side. To indicate the side of the line represented by the inequality, we shall use a little arrow in Figure 1. The shaded area is the region of points satisfying all four inequalities simultaneously. In this case the region is unbounded. If the restraint

$$(14) \quad x_1 + 2x_2 \leq 6$$

is added to the system the common region will become bounded (see Figure 2). If now the restraint

$$(15) \quad -x_1 + 4x_2 \geq 0$$

is added, the region is reduced to a unique point (4, 1) (see Figure 3). Finally, if the condition

$$(16) \quad -2x_1 + x_2 \geq 2$$

is added to the system, no points satisfy all restraints simultaneously and there exist no solutions.

If it is required to find a solution that minimizes the form

$$(17) \quad -2x_1 - x_2 = z,$$

where the point (x_1, x_2) lies in the shaded region (see Figure 4) the line $2x_1 + x_2 = \text{constant}$ is moved parallel to itself until it just touches the shaded area at the extreme right point (6, 0). The (unique) optimal solution is given by $x_1 = 6$, $x_2 = 0$

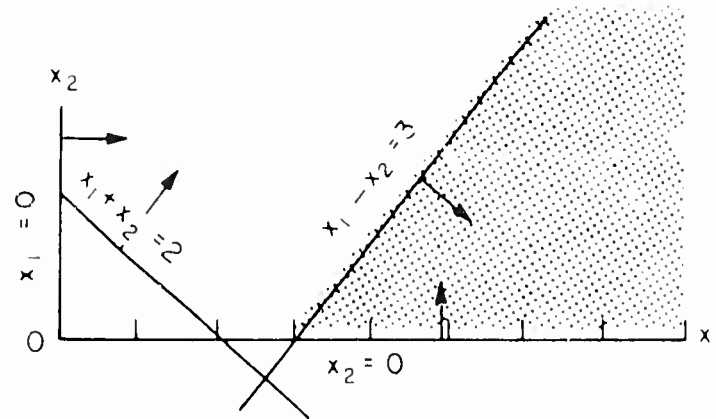


Figure 1

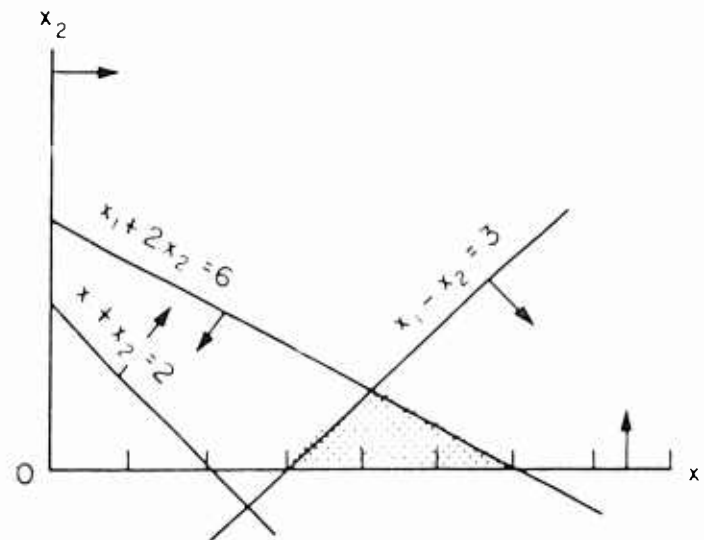


Figure 2

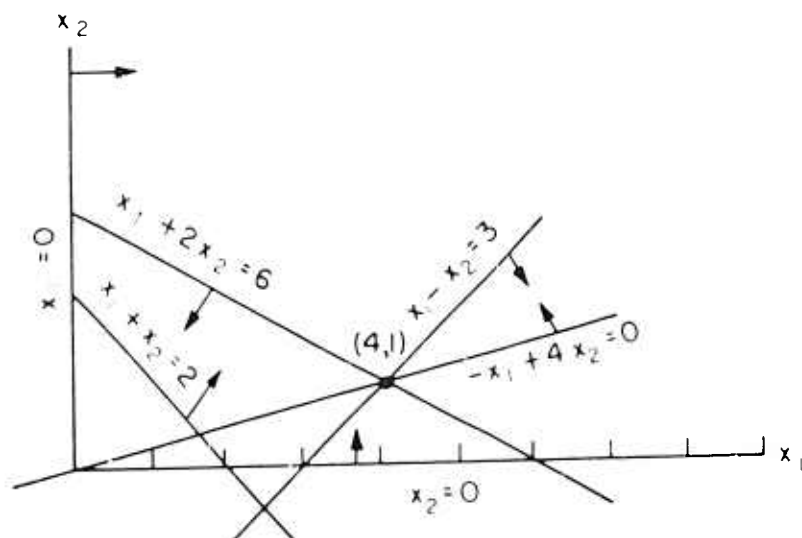


Figure 3

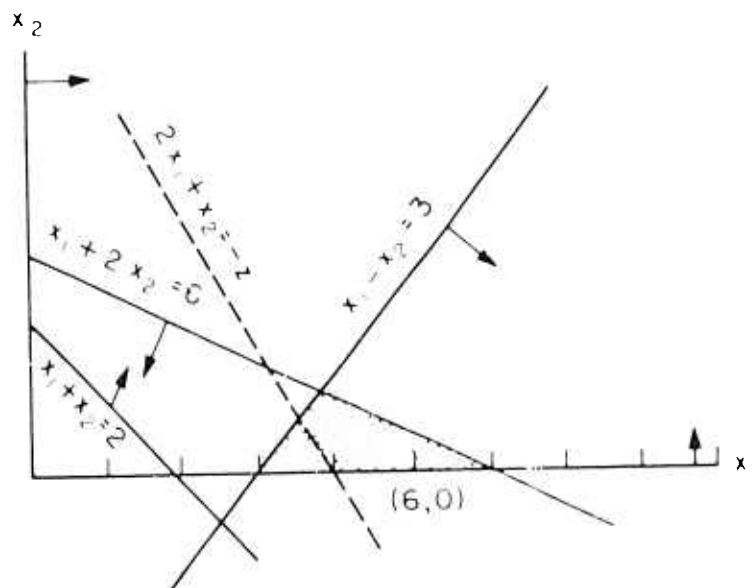


Figure 4

and $z = -2(6) - (0) = -12$. On the other hand, if the problem were to minimize this same form over the shaded region of Figure 1, it would be possible to move the line $2x_1 + x_2 = -z$ indefinitely to the right and still cut the region of solutions. In the latter case, it would be possible to construct solutions such that z can be made smaller than any pre-assigned value.

It will be noted that the minimizing solution is unique in Figure 4 for whatever linear form is chosen, unless the line $z = \text{constant}$ is parallel to one of the sides of the shaded area. In general, the optimal solution is unique unless the line represented by the objective form $z = \text{constant}$ is in a special position relative to the other lines. In the latter case, a slight modification (or perturbation as it is called) of the coefficients in the objective form can bring about uniqueness. Paradoxically, in practical cases, non-uniqueness is rather the rule than the exception. The underlying reason for this is not clear. Mathematical models for economic, military, and industrial applications exhibit very special structures, and this is certainly part of the explanation. A second partial reason for this may be that the coefficients a_{ij} and c_j in practice are carried with only a few significant figures.

Problem: Show that the values of x_1 and x_2 that yield the maximum value of the form $-x_1 - 2x_2 = z$ are not unique in Figure 4. Show, by slightly perturbing the coefficients of the form, that either $(6, 0)$ or $(4, 1)$ will yield the unique minimum.

Where non-uniqueness is the rule rather than the exception, it is a good thing to have at hand a second criterion, such as an alternative objective form. After minimization of the first form z , the equation $z = z_0$, where $z_0 = \text{Min } z$, is added to the system of equations and the second form is then, in turn, minimized.

PROBLEMS

1. Assuming that firing is the opposite of hiring, give reasons why it is better to treat this as two non-negative activities rather than as a single activity with positive and negative activity levels.
2. If an activity such as steel production requires capital such as bricks and cement to build blast furnaces, what would the negative of this activity imply if it were used as an admissible activity?
3. Suppose that the difference between production and requirements is interpreted as surplus or deficit (depending on sign). Illustrate how surplus can be interpreted as a storage activity and deficit as a purchasing activity in which all coefficients of the associated variables can be quite different.
4. Reduce system

$$(a) \quad x_1 + x_2 \geq 2$$

$$x_1 - x_2 \leq 4$$

$$x_1 + x_2 \leq 7$$

$$(b) \quad x_1 + x_2 \geq 2$$

$$x_1 - x_2 \leq 4$$

$$x_1 + x_2 + x_3 = 7$$

to a system of equalities in non-negative variables by two different methods. Show that systems (a) and (b) correspond to cases (a) and (b) of the alternative method.

5. Reduce the same system as above where it is assumed also $x_1 \geq 0, x_2 \geq 0$.

6. Suppose steaks contain per unit 1 unit of carbohydrates, 3 units of vitamins, 3 units of proteins and cost 50 units of cash. Suppose potatoes per unit contain 3, 4, 1, and 25 units of these items respectively. Letting x be the quantity of steaks and y the quantity of potatoes, express the mathematical relations that must be satisfied to meet the minimum requirements of 8 units of carbohydrates, 19 units of vitamins, and 7 units of protein. If x and y are to be chosen so that the cost of diet is a minimum, what is the objective form?
7. Reduce the inequality system of problem six to an equality system in non-negative variables.
8. Using x and y as coordinates of a point in two-space, plot the various relations of problem 6 and describe the set of points which constitute the set of feasible solutions; what is the optimal solution? Is it unique?
9. Reduce the system

$$\begin{aligned}x_1 + x_2 + x_3 &= 5 & x_1 \geq 0, x_2 \geq 0 \\x_1 - x_2 + x_3 &= 7 \\x_1 + 2x_2 + 4x_3 &\leq 2\end{aligned}$$

to an equivalent inequality system.

10. Solve graphically the system in non-negative variables:

$$\begin{aligned}x_1 + x_2 &\leq 1 \\4x_1 + 8x_2 &\leq 32 \\x_1 + x_2 &\leq 4 \\x_1 - 2x_2 &\geq 2.\end{aligned}$$

What inequalities are implied by others?

11. Transform the system below into a system of equations in non-negative variables by two methods:

$$2x_1 + 3x_2 + 4x_3 \geq 5$$

$$4x_1 - 7x_2 + 3x_3 \leq 4.$$

12. Transform the system of equations below in non-negative variables into a system of inequalities by two methods:

$$2x_1 + 3x_2 + 4x_3 = 5$$

$$4x_1 - 7x_2 + 3x_3 = 4.$$